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# Convergence Properties Related to $p$-Point Padé Approximants of Stieltjes Transforms

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Let $\psi$ be a finite positive measure on $\mathbf{R}$, and let $F_\psi(z)=\int_{-\infty}^{\infty}(d\psi(t)/(z-t))$ be its Stieltjes transform. A special multipoint Padé approximation problem for $F_\psi(z)$ is studied, where the interpolation points are a finite number of points $a_1, ..., a_p$ in $\mathbf{R}$ repeated cyclically and the support of $\psi$ is contained in an interval bounded by adjacent interpolation points. For the case $p=3$ monotone convergence of each of the subsequences $\{P_{3q+m}(z)/Q_{3q+m}(z)\}$, $m=0, 1, 2$, of the multipoint Padé approximants $\{P_n(z)/Q_n(z)\}$ is established, and sufficient conditions (involving general moments $c_j^{(i)}=\int_{-\infty}^{\infty}(d\psi(t)/(t-a_i)^j)$) for divergence of the series $\sum_{q=1}^{\infty}|Q_{3q+m}(z)|^2$ are given. © 1993 Academic Press, Inc.

## 1. INTRODUCTION

By a distribution we mean a finite positive measure $\psi$ on $\mathbf{R}$ with infinite support. By its Stieltjes transform we mean the function $F_\psi(z)=\int_{-\infty}^{\infty}(d\psi(t)/(z-t))$.

Let $a_1, a_2, ..., a_p$ be distinct points in $\mathbf{R}$, $a_1<a_2<\cdots<a_p$. We call the sets $[a_1, a_2]$, $[a_2, a_3]$, ..., $[a_{p-1}, a_p]$, $(-\infty, a_1]\cup[a_p, \infty)$ *Stieltjes intervals* for the point set $\{a_1, ..., a_p\}$. We call the distribution function $\psi$ a *Stieltjes distribution* for the point set $\{a_1, ..., a_p\}$ if the support $S(\psi)$ is contained in a Stieltjes interval I, and all the moments

$$c_0=\int_I d\psi(t), \qquad c_j^{(i)}=\int_I \frac{d\psi(t)}{(t-a_i)^j} \tag{1.1}$$
$$i=1, ..., p, \qquad j=1, 2, ...,$$

exist. Note that $S(\psi)$ may contain one or two (or none) of the points in $\{a_1, ..., a_p\}$.

We later prove (in the case $p=3$) that when $\psi$ has its support in one Stieltjes interval I, then certain subsequences of multipoint Padé



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approximants converge monotonically on the remaining (open) Stieltjes intervals.

When $c_0$, $c_j^{(i)}$, and a Stieltjes interval $I$ are given, the extended Stieltjes moment problem (ESMP) consists of finding distributions $\psi$ with $S(\psi) \subset I$ such that Eqs. (1.1) are satisfied. Conditions for existence and uniqueness of solutions of the ESMP were discussed in [17]. Properties equivalent to unique solvability were also derived. However, it was not observed that these conditions are always satisfied when $p \geqslant 3$, since $F_\psi$ is holomorphic at the points $a_i$ outside the Stieltjes interval $I$.

For each natural number $n$ we let $r = r_n$ be defined by $1 \leqslant r \leqslant p$, $n = r \pmod{p}$, and we let $q = q_n$ be defined as the integer part $[n/p]$ of $n/p$. Thus $n = qp + r$. We also write $a_n$ for $a_{r_n}$.

In the following, $\psi$ is a given distribution. The Stieltjes transform $F_\psi$ has the formal power series expansions

$$\begin{aligned} F_\psi(z) &\approx \sum_{j=0}^{\infty} -c_{j+1}^{(i)}(z-a_i)^j \qquad \text{at} \quad a_i, i = 1, ..., p \\ F_\psi(z) &= \frac{c_0}{z} + O(z^{-1}) \qquad \text{at } \infty. \end{aligned} \tag{1.2}$$

By the $(n-1, n)$ $p$-point Padé approximant for $F_\psi$ we mean the (unique) rational function $U_n(z)/V_n(z)$ with $\deg U_n \leqslant n-1$, $\deg V_n = n$, which interpolates $F_\psi$ in the sense

$$\begin{aligned} U_n(z)/V_n(z) - \frac{c_0}{z} &= O\left(\left[\frac{1}{z}\right]^2\right) \\ U_n(z)/V_n(z) - \sum_{j=0}^{s} -c_{j+1}^{(i)}(z-a_i)^j &= O([z-a_i]^{s+1}), \end{aligned} \tag{1.3}$$

where $s = 2q+1$ for $i < r$, $s = 2q$ for $i = r$, and $s = 2q-1$ for $i > r$.

(Here and in the following, obvious modifications of notation are necessary when indices $r+k$, $r-k$ appear where $r+k > p$, $r-k < 0$. The $p$-point Padé approximants are special cases of multipoint Padé approximants (MPA). For existence of the MPA, see Section 2.)

For general information on multipoint Padé approximants, see, e.g., [1-3, 8, 10-12, 18, 20].

Let $S(\psi) \subset I$, $I$ a Stieltjes interval. In this paper we discuss a general method for obtaining monotonic convergence of subsequences on intervals $J^0$, where $J$ are Stieltjes intervals disjoint from $I$. The methods of proof do not work when the suppert of $\psi$ is only contained in some union (not a single) of Stieltjes intervals. We give specific results for the case $p = 3$, and use these to obtain conditions for divergence of subseries of the series $\sum_{n=1}^{\infty} |Q_n(z)|^2$, where $Q_n(z)$ are the normalized orthogonal rational functions associated with the approximation problem (see Section 2).

Our approach is based on arguments employed by Karlsson and von Sydow [9] in the ordinary Padé approximation situation (i.e., interpolation at $\infty$). See also [4] where the ordinary two-point situation (i.e., interpolation at $\infty$ and 0) is discussed.

## 2. Orthogonal Rational Functions and MPA

Let $\mathscr{R} = \mathscr{R}(\{a_1, ..., a_p\})$ denote the linear space of all rational functions with no poles in the extended complex plane $\hat{\mathbf{C}}$ outside $\{a_1, .., a_p\}$. Thus $\mathscr{R}$ consists of all functions of the form

$$R(z) = \alpha_0 + \sum_{i=1}^{p} \sum_{j=1}^{n_i} \frac{\alpha_{ij}}{(z-a_i)^j}, \qquad \alpha_0, \alpha_{ij} \in \mathbf{C}. \tag{2.1}$$

The Stieltjes distribution $\psi$ defines an inner product $\langle \, , \rangle$ on $\mathscr{R}$ by

$$\langle R, S \rangle = \int_I R(t) \overline{S(t)} \, d\psi(t). \tag{2.2}$$

By applying the Gram–Schmidt procedure to the sequence

$$\left\{1, \frac{1}{z-a_i}, ..., \frac{1}{z-a_p}, \frac{1}{(z-a_1)^2}, ..., \frac{1}{(z-a_p)^2}, \frac{1}{(z-a_1)^3}, ...\right\}$$

we obtain a monic orthogonal sequence $\{S_n\}$ in $\mathscr{R}$. The functions $S_n$ may be written as

$$S_n(z) = V_n(z)/N_n(z), \tag{2.3}$$

where

$$N_n(z) = (z-a_1)^{q+1} \cdots (z-a_r)^{q+1} (z-a_{r+1})^q \cdots (z-a_p)^q \tag{2.4}$$

and $V_n$ is a polynomial of degree $n$. (For the meaning of $q$ and $r$, see Section 1.) We may also write

$$S_n(z) = \beta_0^{(n)} + \frac{\beta_1^{(n)}}{z-a_1} + \cdots + \frac{\beta_p^{(n)}}{z-a_p} + \cdots + \frac{\beta_{n-1}^{(n)}}{(z-a_{r-1})^{q+1}} + \frac{1}{(z-a_r)^{q+1}}. \tag{2.5}$$

The associated functions $R_n$ are defined by

$$R_n(z) = \int_I \frac{S_n(t) - S_n(z)}{t - z}\, d\psi(t). \tag{2.6}$$

The functions $R_n$ belong to $\mathscr{R}$ and may be written in the form

$$R_n(z) = U_n(z)/N_n(z), \tag{2.7}$$

where $U_n$ is a polynomial of degree at most $n-1$. We denote by $P_n$ and $Q_n$ the normalized functions, i.e.,

$$P_n(z) = \| S_n \|^{-1} R_n(z), \qquad Q_n(z) = \| S_n \|^{-1} S_n(z). \tag{2.8}$$

The rational functions $U_n(z)/V_n(z) = R_n(z)/S_n(z)$ are MPAs for $F_\psi$ in the sense that Eqs. (1.3) are satisfied. (See [15].)

The error term $E_n(z, \psi) = R_n(z)/S_n(z) - F_\psi(z)$ may be written in the form.

$$E_n(z, \psi) = \frac{1}{S_n(z)} \int_I \frac{S_n(t)}{t - z}\, d\psi(t) \tag{2.9}$$

$$E_n(z, \psi) = \frac{1}{(z - a_r)\, S_n(z)^2} \int_I \frac{(t - a_r)\, S_n(t)^2}{t - z}\, d\psi(t). \tag{2.10}$$

(See [17].)

The zeros of $S_n$ are simple and lie in $I$. (See [16]. A slight modification of the argument given there shows that this result holds also when $I = (-\infty, a_1] \cup [a_p, \infty)$.) It follows in particular that the coefficient $\beta_{n-1}^{(n)}$ in (2.5) is different from zero.

We make use of the fact that the functions $R_n(z)/S_n(z)$ are the approximants of a continued fraction $K_{n=1}^{\infty}(a_n(z)/b_n(z))$, where the elements have the form

$$a_1(z) = \frac{C_1}{z - a_1}, \qquad a_2(z) = \frac{C_2}{z - a_2},$$
$$a_n(z) = \frac{C_n(z - a_{n-2})}{z - a_n} \quad \text{for} \quad n \geqslant 3, \tag{2.11}$$

$$b_1(z) = \frac{A_1 + B_1(z - a_1)}{z - a_1}, \qquad b_2(z) = \frac{A_2(z - a_1) + B_2}{z - a_2},$$
$$b_n(z) = \frac{A_n(z - a_{n-1}) + B_n(z - a_{n-2})}{z - a_n} \quad \text{for} \quad n \geqslant 3. \tag{2.12}$$

Here the coefficients $A_n$, $B_n$, $C_n$ are given by

$$A_1 = 1, \qquad A_2 = \frac{\beta_0^{(2)}}{\beta_0^{(1)}},$$
$$A_n = \frac{\beta_{n-2}^{(n)}(a_n - a_{n-2})}{\beta_{n-2}^{(n-1)}(a_{n-1} - a_{n-2})} \qquad \text{for} \quad n \geqslant 3, \tag{2.13}$$

$$B_1 = \beta_0^{(1)}, \qquad B_2 = -\beta_1^{(2)}(a_2 - a_1),$$
$$B_n = \frac{-\beta_{n-1}^{(n)}(a_n - a_{n-1})}{a_{n-1} - a_{n-2}} \qquad \text{for} \quad n \geqslant 3, \tag{2.14}$$

$$C_1 = \beta_0^{(1)} c_0, \qquad C_2 = \frac{-\beta_1^{(2)}(a_2 - a_1) \, \| S_1 \|^2}{\beta_0^{(1)} c_0},$$
$$C_n = \frac{-\beta_{n-1}^{(n)}(a_n - a_{n-1}) \, \| S_{n-1} \|^2}{\beta_{n-2}^{(n-1)}(a_{n-1} - a_{n-2}) \, \| S_{n-2} \|^2} \qquad \text{for} \quad n \geqslant 3. \tag{2.15}$$

We recall that

$$\begin{bmatrix} R_n(z) \\ S_n(z) \end{bmatrix} = b_n(z) \begin{bmatrix} R_{n-1}(z) \\ S_{n-1}(z) \end{bmatrix} + a_n(z) \begin{bmatrix} R_{n-2}(z) \\ S_{n-2}(z) \end{bmatrix}$$
$$\text{for} \quad n \geqslant 1, R_{-1} = 1, R_0 = 0, S_{-1} = 0, S_0 = 1. \tag{2.16}$$

For these properties of $R_n$, $S_n$, see [6]. For basic information on continued fractions, see, e.g., [7].

The zeros $t_1, ..., t_n$ of $S_n$ are nodes for a quadrature formula with positive weights $\lambda_1, ..., \lambda_n$ which is exact for all functions $R$ of the form (2.1) with $n_i \leqslant 2q + 2$ for $i < r$, $n_r \leqslant 2q + 1$, $n_i \leqslant 2q$ for $i > r$. Thus for such functions we have

$$\int_F R(t) \, d\psi(t) = \sum_{k=1}^{n} \lambda_k R(t_k). \tag{2.17}$$

(See [13, 14].) This result applied to the function $f(t) = (S_n(t) - S_n(z))/(t - z)$ yields the formula

$$R_n(z)/S_n(z) = \sum_{k=1}^{n} \frac{\lambda_k}{z - t_k}. \tag{2.18}$$

Since $\sum_{k=1}^{n} \lambda_k = c_0$, it follows that the sequence $\{R_n(z)/S_n(z)\}$ is uniformly bounded on every compact subset of $\mathbf{C} - I$.

We define the step function $\psi_n$ by $\psi_n(t) = \sum \{\lambda_k : t \leqslant t_k\}$. Then $0 \leqslant \psi_n(t) \leqslant c_0$ for all $n$ and $t$. By Helly's theorems it follows that every subsequence of $\{\psi_n\}$ contains a subsequence $\{\psi_{n(\nu)}\}$ which converges to a solution $\varphi$ of the ESMP associated with the moments (1, 1), and $R_{n(\nu)}(z)/S_{n(\nu)}(z) = \int_I (d\psi_{n(\nu)}(t)/(z-t))$ converges to $F_\psi(z)$ for $z \in \mathbf{C} - I$. It easily follows that if the ESMP has $\psi$ as its unique solution, then $\{R_n(z)/S_n(z)\}$ converges to $F_\psi(z)$, locally uniformly on $\mathbf{C} - I$.

Let $a_i \notin I$. Then for all solutions $\psi$, $F_\psi$ have the same power series expansions at $a_i$, hence all $F_\psi$ are equal. It follows that for $p \geqslant 3$ the ESMP has a unique solution, and hence $\{R_n(z)/S_n(z)\}$ converges to $F_\psi$.

In Section 4 we establish monotonic convergence of certain subsequences of $\{R_n(z)/S_n(z)\}$. In the proof it will not be made use of the fact (obtained by normal families arguments) that the whole sequence $\{R_n(z)/S_n(z)\}$ is convergent.

For more detailed information about the functions $R_n$, $S_n$, $R_n/S_n$ we refer to [13–17]. The necessary results on convergence of analytic functions can be found, e.g., in [19].

## 3. Conditions on Subsequences of MPA

Let $p, q \in \mathbf{N}$, $p > q$. We write

$$D[p, q] = D[p, q, z] = R_p(z)/S_p(z) - R_q(z)/S_q(z). \tag{3.1}$$

In the following, $\{n(\nu)\}$ denotes a subsequence of $\mathbf{N}$. Recall that $Q_n(z) = \|S_n\|^{-1} S_n(z)$.

Theorem 3.1. *Let* $z \in \mathbf{R}$. *Assume that the following conditions are satisfied*:

$$\sum_{\nu=1}^{\infty} |D[n(\nu+1), n(\nu), z]| < \infty \tag{3.2}$$

$$\sum_{\nu=1}^{\infty} |D[n(\nu+1), n(\nu), z]\, Q_{n(\nu+1)}(z)\, Q_{n(\nu)}(z)|^{1/2} = \infty. \tag{3.3}$$

*Then* $\sum_{n=1}^{\infty} |Q_{n(\nu)}(z)|^2 = \infty$.

*Proof.* By the Schwartz inequality we obtain

$$\sum_{\nu=1}^{\infty} |D[n(\nu+1), n(\nu), z]\, Q_{n(\nu+1)}(z)\, Q_{n(\nu)}(z)|^{1/2} \leqslant \left\{\sum_{\nu=1}^{\infty} |D[n[\nu+1), n(\nu), z]|\right\}^{1/2} \cdot \left\{\sum_{\nu=1}^{\infty} |Q_{n(\nu+1)}(z)\, Q_{n(\nu)}(z)|\right\}^{1/2}. \tag{3.4}$$

From (3.2)–(3.3) we then obtain $\sum_{\nu=1}^{\infty} |Q_{n(\nu+1)}(z)\, Q_{n(\nu)}(z)| = \infty$, hence also

$$\sum_{\nu}^{\infty} |Q_{n(\nu)}(z)|^2 = \infty. \quad \blacksquare \tag{3.5}$$

*Remark.* We note that in order to establish (3.2) it suffices to show that the sequence $\{R_{n(\nu)}(z)/S_{n(\nu)}(z)\}$ is bounded and monotonic.

## 4. Bounded Monotonic Subsequences of MPA for $p = 3$

In this section we study the subsequences $\{R_{3q+1}/S_{3q+1}\}$, $\{R_{3q+2}/S_{3q+2}\}$, and $\{R_{3q+3}/S_{3+3}\}$ in the case that $p = 3$. Thus we have three points $a_1, a_2, a_3$ where $a_1 < a_2 < a_3$, and $S(\psi) \subset I$ where $I$ is one of the sets $[a_1, a_2]$, $[a_2, a_3]$, $(-\infty, a_1] \cup [a_3, \infty)$.

By repeated use of the recurrence relations (2.16) we obtain the formulas

$$D[n, n-1, z]\, S_n(z)\, S_{n-1}(z) = (-1)^{n-1} a_1(z) \cdots a_n(z) \tag{4.1}$$

$$D[n, n-2, z]\, S_n(z) = S_{n-1}(z)\, b_n(z)\, D[n-1, n-2, z] \tag{4.2}$$

$$D[n, n-3, z]\, S_n(z) = S_{n-2}(z)[b_n(z)\, b_{n-1}(z) + a_n(z)] \times D[n-2, n-3, z]. \tag{4.3}$$

Substitution from the formulas (2.11)–(2.15) yields (for $n \geqslant 5$)

$$\begin{aligned} D[n, n-3, z]\, S_n(z)\, S_{n-3}(z) &= [\{A_n(z-a_{n-1}) + B_n(z-a_{n-2})\}\{A_{n-1}(z-a_{n-2}) + B_{n-1}(z-a_{n-3})\} \\ &\quad + C_n(z-a_{n-1})(z-a_{n-2})]\, \frac{D[n-2, n-3, z]\, S_{n-2}(z)\, S_{n-3}(z)}{(z-a_n)(z-a_{n-1})} \end{aligned} \tag{4.4}$$

and (when $p = 3$)

$$D[n, n-3, z] = \frac{\|S_{n-3}\|^2\, \Gamma_n(z)}{S_n(z)\, S_{n-3}(z)}, \tag{4.5}$$

where

$$\Gamma_n(z) = \gamma_{n,1} + \gamma_{n,2} + \gamma_{n,3} + \gamma_{n,4} + \gamma_{n,5} \tag{4.6}$$

and

$$\gamma_{n,1} = \frac{-\beta_{n-2}^{(n)}\beta_{n-3}^{(n-1)}(a_n - a_{n-1})(a_n - a_{n-2})}{\beta_{n-2}^{(n-1)}(a_{n-1} - a_{n-2})(z - a_n)^2} \tag{4.7a}$$

$$\gamma_{n,2} = \frac{\beta_{n-1}^{(n)}\beta_{n-3}^{(n-1)}(a_n - a_{n-1})^2 (z - a_{n-2})}{(a_{n-1} - a_{n-2})(z - a_{n-1})(z - a_n)^2} \tag{4.7b}$$

$$\gamma_{n,3} = \frac{-\beta_{n-2}^{(n)}\beta_{n-3}^{(n-2)}(a_n - a_{n-2})}{(z - a_{n-2})(z - a_n)} \tag{4.7c}$$

$$\gamma_{n,4} = \frac{-\beta_{n-1}^{(n)}\beta_{n-2}^{(n-1)}\beta_{n-3}^{(n-2)}(a_n - a_{n-1})}{(z - a_{n-1})(z - a_n)} \tag{4.7d}$$

$$\gamma_{n,5} = \frac{\beta_{n-1}^{(n)} \| S_{n-1} \|^2 (a_n - a_{n-1})(a_n - a_{n-2})}{\beta_{n-2}^{(n-1)} \| S_{n-2} \|^2 (a_{n-1} - a_{n-2})(z - a_n)^2}. \tag{4.7e}$$

In the following, $I$ is one of the intervals $I_1 = [a_1, a_2]$, $I_2 = [a_2, a_3]$, $I_3 = (-\infty, a_1] \cup [a_3, \infty)$.

PROPOSITION 4.1. *Let $S(\psi) \subset I$. The following implications hold:*

(a) *If $I = I_1$, then $D[3q+3, 3q, z] < 0$ for $z \in I_3$, $D[3q+2, 3q-1, z] < 0$ for $z \in I_2$, $D[3q+1, 3q-2, z] > 0$ for $z \in I_2$.*

(b) *If $I = I_2$, then $D[3q+3, 3q, z] < 0$ for $z \in I_3$, $D[3q+2, 3q-1, z] > 0$ for $z \in I_3$, $D[3q+1, 3q-2, z] < 0$ for $z \in I_1$.*

(c) *If $I = I_3$, then $D[3q+3, 3q, z] > 0$ for $z \in I_1$, $D[3q+2, 3q-1, z] < 0$ for $z \in I_2$, $D[3q+1, 3q-2, z] < 0$ for $z \in I_1$.*

*Proof.* We prove (a); the proofs of (b) and (c) are similar.
For $n = 3q+3$ we may write

$$S_n(z) = \frac{1}{(z-a_3)^{q+1}} + \frac{\beta_{n-1}^{(n)}}{(z-a_2)^{q+1}} + \frac{\beta_{n-2}^{(n)}}{(z-a_1)^{q+1}} + \cdots \tag{4.8a}$$

$$S_{n-1}(z) = \frac{1}{(z-a_2)^{q+1}} + \frac{\beta_{n-2}^{(n-1)}}{(z-a_1)^{q+1}} + \frac{\beta_{n-3}^{(n-1)}}{(z-a_3)^{q}} + \cdots \tag{4.8b}$$

$$S_{n-2}(z) = \frac{1}{(z-a_1)^{q+1}} + \frac{\beta_{n-3}^{(n-2)}}{(z-a_3)^{q}} + \frac{\beta_{n-4}^{(n-2)}}{(z-a_2)^{q}} + \cdots. \tag{4.8c}$$

We consider the case that $q$ is even, the argument for odd $q$ is similar. Note that $n$ is odd when $q$ is even.

Since $1/(z-a_3)^{q+1}$ is the dominating term in $S_n(z)$ near $a_3$, $\beta_{n-1}^{(n)}/(z-a_2)^{q+1}$ is the dominating term near $a_2$, and $\beta_{n-2}^{(n)}/(z-a_1)^{q+1}$ is the dominating term near $a_1$, and since all the zeros of $S_n$ lie in $(a_1, a_2)$, we find that $S_n(z)>0$ for $z>a_3$, $S_n(z)<0$ for $z\in(a_2, a_3)$, $S_n(z)>0$ for $z<a_3$. From this it follows that $\beta_{n-1}^{(n)}<0$ and $\beta_{n-2}^{(n)}<0$.

By similar arguments we see that $S_{n-3}(z)>0$ for $z\in I_3$ and that $\beta_{n-2}^{(n-1)}<0$, $\beta_{n-3}^{(n-1)}>0$, $\beta_{n-3}^{(n-2)}<0$, $\beta_{n-4}^{(n-2)}<0$.

It follows that all the terms $\gamma_{n,k}$, $k=1, 2, 3, 4, 5$, are negative for $z\in I_3$, hence $D[3q+3, 3q, z]<0$ for $z\in I_3$.

The arguments for $n=3q+2$ and $n=3q+1$ are similar. ∎

PROPOSITION 4.2. *Let $S(\psi)\subset I$. The following implications hold*:

(a) *If $I=I_1$, then $E_{3q+3}(z,\psi)>0$ for $z\in I_3$, $E_{3q+2}(z,\psi)>0$ for $z\in I_2$, $E_{3q+1}(z,\psi)<0$ for $z\in I_2$.*

(b) *If $I=I_2$, then $E_{3q+3}(z,\psi)>0$ for $z\in I_3$, $E_{3q+2}(z,\psi)<0$ for $z\in I_3$, $E_{3q+1}(z,\psi)>0$ for $z\in I_1$.*

(c) *If $I=I_3$, then $E_{3q+3}(z,\psi)<0$ for $z\in I_1$, $E_{3q+2}(z,\psi)>0$ for $z\in I_2$, $E_{3q+1}(z,\psi)>0$ for $z\in I_2$.*

*Proof.* This follows easily from (2.10). ∎

THEOREM 4.3. *Let $I=I_k$ be one of the intervals $I_1, I_2, I_3$, and let $S(\psi)\subset I$. Then each of the subsequences $\{R_{3q+3}/S_{3q+3}\}$, $\{R_{3q+2}/S_{3q+2}\}$, $\{R_{3q+1}/S_{3q+1}\}$ converges monotonically to a limit in each of the two remaining open intervals $I_j$, $j\neq k$.*

*Proof.* This follows immediately from Proposition 4.1. and Proposition 4.2. ∎

*Remark.* We recall that Theorem 4.3 implies that each of the subsequences in that theorem satisfies the condition (3.2) of Theorem 3.1 (with $v$ replaced by $q$).

## 5. CARLEMAN TYPE CONDITIONS FOR SUBSERIES OF $\sum_{n=1}^{\infty}|Q_n(z)|^2$

As in Section 4 we consider the situation that $p=3$, $a_1<a_2<a_3$, and $S(\psi)\subset I$ where $I$ is one of the Stieltjes intervals $I_1=[a_1, a_2]$, $I_2=[a_2, a_3]$, $I_3=(-\infty, a_1]\cup[a_3, \infty)$.

PROPOSITION 5.1. *The following inequalities hold for $z\notin\{a_1, a_2, a_3\}$*: (a) $\liminf_q |\Gamma_{3q+1}(z)|>0$, (b) $\liminf_q |\Gamma_{3q+2}(z)|>0$, (c) $\liminf_q |\Gamma_{3q}(z)|>0$.

*Proof.* Let $t_1^{(n)}, ..., t_n^{(n)}$ be the zeros of $S_n(z)$. We see from (2.3)–(2.5) that we may write

$$S_n(z)=\frac{\beta_0^{(n)}(z-t_1^{(n)})\cdots(z-t_n^{(n)})}{N_n(z)}. \tag{5.1}$$

Comparison of (2.5) and (5.1) gives

$$\beta_0^{(n)} = \frac{\prod_{m \neq r} (a_r - a_m)^{s_m}}{\prod_{m=1}^{n} (a_r - t_m^{(n)})}, \tag{5.2}$$

where $s_m = q + 1$ for $m \leqslant r$, $s_m = q$ for $m > r$.

We also obtain

$$\beta_{n-1}^{(n)} = \beta_0^{(n)} \frac{\prod_{m=1}^{n} (a_{r-1} - t_m^{(n)})}{\prod_{m \neq r-1} (a_{r-1} - a_m)^{s_m}} \tag{5.3}$$

hence also

$$\beta_{n-1}^{(n)} = \frac{\prod_{m=1}^{n} (a_{r-1} - t_m^{(n)}) \cdot \prod_{m \neq r} (a_r - a_m)^{s_m}}{\prod_{m=1}^{n} (a_r - t_m^{(n)}) \cdot \prod_{m \neq r-1} (a_{r-1} - a_m)^{s_m}}. \tag{5.4}$$

Similarly we get

$$\beta_{n-2}^{(n)} = \frac{\prod_{m=1}^{n} (a_{r-2} - t_m^{(n)}) \cdot \prod_{m \neq r} (a_r - a_m)^{s_m}}{\prod_{m=1}^{n} (a_r - t_m^{(n)}) \prod_{m \neq r-2} (a_{r-2} - a_m)^{s_m}} \tag{5.5}$$

(recall that $a_{r-3} = a_r$).

From these formulas follow

$$\frac{\beta_{n-2}^{(n)} \beta_{n-3}^{(n-1)}}{\beta_{n-2}^{(n-1)}} = \frac{\prod_{m=1}^{n} (a_{r-2} - t_m^{(n)}) \prod_{m=1}^{n-1} (a_r - t_m^{(n-1)}) \prod_{m=1}^{n-1} (a_{r-1} - t_m^{(n-1)})}{\prod_{m=1}^{n} (a_r - t_m^{(n)}) \prod_{m=1}^{n-1} (a_{r-1} - t_m^{(n-1)}) \prod_{m=1}^{n-1} (a_{r-2} - t_m^{(n-1)})} \tag{5.6}$$

$$\beta_{n-1}^{(n)} \beta_{n-3}^{(n-1)} = \frac{\prod_{m=1}^{n} (a_{r-1} - t_m^{(n)}) \prod_{m=1}^{n-1} (a_r - t_m^{(n-1)})}{\prod_{m=1}^{n} (a_r - t_m^{(n)}) \prod_{m=1}^{n-1} (a_{r-1} - t_m^{(n-1)})}. \tag{5.7}$$

The zeros of $S_{n-1}$ separate those of $S_n$ (see [16]), hence

$$\left| \frac{\beta_{n-2}^{(n)} \beta_{n-3}^{(n-1)}}{\beta_{n-2}^{(n-1)}} \right| \geqslant \frac{|a_{r-2} - t_0^{(n)}|}{|a_r - t_{00}^{(n)}|} \tag{5.8}$$

where $t_0^{(n)}$ and $t_{00}^{(n)}$ are $t_1^{(n)}$ or $t_n^{(n)}$.

Similarly

$$|\beta_{n-1}^{(n)} \beta_{n-3}^{(n-1)}| \geqslant \frac{|a_{r-1} - t_0^{(n)}|}{|a_r - t_{00}^{(n)}|} \tag{5.9}$$

and

$$|\beta_{n-1}^{(n)} \beta_{n-2}^{(n-1)} \beta_{n-3}^{(n-2)}| \geqslant \left| \frac{(a_{r-1} - t_0^{(n)})(a_{r-2} - t_0^{(n-1)})}{(a_r - t_n^{(n)})(a_r - t_0^{(n)})} \right|. \tag{5.10}$$

Now assume $I=I_1$. If $n=3q+2$ then $r=2$. Since all the zeros of $S_n$ lie in $I_1$ it follows from (5.8) that there is a positive constant $A$ such that

$$\left|\frac{\beta_{3q}^{(3q+2)}\beta_{3q-1}^{(3q+1)}}{\beta_{3q}^{(3q+1)}}\right| \geqslant A \qquad \text{for all } q, \tag{5.11}$$

Similarly when $n=3q+1$, then $r=1$, and it follows from (5.9) that there is a positive constant $B$ such that

$$|\beta_{3q}^{(3q+1)}\beta_{3q-2}^{(3q)}| \geqslant B \qquad \text{for all } q, \tag{5.12}$$

and if $n=3q+3$ it follows from (5.10) that there is a positive constant $C$ such that

$$|\beta_{3q+2}^{(3q+3)}\beta_{3q+1}^{(3q+2)}\beta_{3q}^{(3q+1)}| \geqslant C \qquad \text{for all } q. \tag{5.13}$$

Since all the terms $\gamma_{n,k}$, $k=1, 2, 3, 4, 5$, have the same sign, it follows that $\liminf_q |\Gamma_{3q+2}(z)|>0$, $\liminf_q |\Gamma_{3q+1}(z)|>0$, and $\liminf_q |\Gamma_{3q}(z)|>0$.

The arguments in cases $I=I_2$ and $I=I_3$ are similar. (Note that if $I=I_3$ and $\lim_n t_n^{(n)}=\infty$ or $\lim_n t_1^{(n)}=-\infty$, then also (5.4)–(5.5) is used directly.) ∎

**Proposition** 5.2. *Let* $S(\psi)\subset I$ *and* $z\in \mathbf{R}-I, z\notin \{a_1, a_2, a_3\}$. *The following implications hold*:

(a) *If* $\displaystyle\sum_{q=1}^{\infty}\left(\frac{\|S_{3q}\|}{\|S_{3q+3}\|}\right)^{1/2}=\infty$ *then* $\displaystyle\sum_{q=1}^{\infty}|Q_{3q}(z)|^2=\infty.$ (5.14)

(b) *If* $\displaystyle\sum_{q=1}^{\infty}\left(\frac{\|S_{3q-1}\|}{\|S_{3q+2}\|}\right)^{1/2}=\infty$ *then* $\displaystyle\sum_{q=1}^{\infty}|Q_{3q+2}(z)|^2=\infty.$ (5.15)

(c) *If* $\displaystyle\sum_{q=1}^{\infty}\left(\frac{\|S_{3q-2}\|}{\|S_{3q+1}\|}\right)^{1/2}=\infty$ *then* $\displaystyle\sum_{q=1}^{\infty}|Q_{3q+1}(z)|^2=\infty.$ (5.16)

*Proof.* Assume, e.g., that (5.14) is satisfied. By Theorem 4.3 the condition (3.2) of Theorem 3.1 is satisfied when $n(q)=3q+3$.

It follows from (4.5) that

$$|D[3q+3, 3q, z]\, Q_{3q+3}(z)\, Q_{3q}(z)| \geqslant \frac{\|S_{3q}\|}{\|S_{3q+3}\|}\,|\Gamma_{3q+3}(z)|. \tag{5.17}$$

So by (5.14) and Proposition 5.1 we conclude that the condition (3.3) of Theorem 3.1 is satisfied.

Similarly the conditions (3.2) and (3.3) of Theorem 3.1 follow from (5.15) and from (5.16).

The conclusion now follows from Theorem 3.1. ∎

THEOREM 5.3. *Let $S(\psi)\subset I$, $z\in\mathbf{R}-I$. The following implications hold: If $\sum_{q=1}^{\infty}|c_q^{(j)}|^{-1/2q}=\infty$, then $\sum_{q=1}^{\infty}|Q_{3q+j}(z)|^2=\infty$, for $j=0,1,2$.*

*Proof.* It follows from Proposition 5.2 that it suffices to show that $\sum_{q=1}^{\infty}|c_q^{(j)}|^{-(1/2)q}=\infty$ implies $\sum_{q=1}^{\infty}(\|S_{3q-3+j}\|/\|S_{3q+j}\|)^{1/2}=\infty$, $j=0,1,2$.

We note that

$$\|S_{3q}\|^2=\int_I\frac{1}{(t-a_3)^q}S_{3q}(t)\,d\psi(t), \tag{5.18}$$

hence by the Schwartz inequality

$$\|S_{3q}\|^2\leqslant\left[\int_I\frac{d\psi(t)}{(t-a_3)^{2q}}\right]^{1/2}\cdot\left[\int_I S_{3q}(t)^2\,d\psi(t)\right]^{1/2}=[c_{2q}^{(3)}]^{1/2}\cdot\|S_{3q}\|. \tag{5.19}$$

Thus

$$\|S_{3q}\|^{-1}\geqslant[c_{2q}^{(3)}]^{-1/2}. \tag{5.20}$$

By Carleman's inequality (see, e.g., [5]) and (5.18) we then get

$$\sum_{q=1}^{\infty}\left(\frac{\|S_{3q}\|}{\|S_{3q+3}\|}\right)^{1/2}\geqslant\frac{1}{e}\sum_{q=1}^{\infty}\left(\frac{1}{\|S_{3q+3}\|}\right)^{1/2(q+1)}\geqslant\frac{1}{e}\sum_{q=1}^{\infty}[c_{2(q+1)}^{(3)}]^{-1/4(q+1)}. \tag{5.21}$$

The sequences $\sum_{q=1}^{\infty}[c_{2(q+1)}^{(3)}]^{-1/4(q+1)}$ and $\sum_{q=1}^{\infty}[c_{q+1}^{(3)}]^{-1/2(q+1)}$ diverge simultaneously (by an argument similar to that in [9, p. 50]). Thus $\sum_{q=1}^{\infty}|c_q^{(3)}|^{-1/2q}=\infty$ implies $\sum_{q=1}^{\infty}(\|S_{3q-3}\|/\|S_{3q}\|)^{1/2}=\infty$.

The arguments for $j=1,2$ are similar. ∎

*Remark.* It can easily be verified that when $I=I_k$, then always $\sum_{q=1}^{\infty}|c_q^{(k+2)}|^{-1/2q}=\infty$. It follows from Theorem 5.3 that then $\sum_{q=1}^{\infty}|Q_{3q+k+2}(z)|^2=\infty$ for $z\notin I$. Thus if $I=I_k$ and $z\notin I_k$ the series $\sum_{q=1}^{\infty}|Q_{3q+k+2}(z)|^2$ always diverges.

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## REFERENCES

1. K. C. CHUI, Recent results on Padé approximants and related problems, *in* "Approximation Theory, II" (C. C. Lorentz, K. C. Chui, and L. L. Schumacher, Eds.), pp. 79–115, Academic Press, Orlando, FL, 1976.

2. J. GELFGREN, Multipoint Padé approximants converging to functions of Stieltjes type, *in* "Padé Approximation and Its Applications" (M. G. de Bruin and H. van Rossum, Eds.), pp. 197–207, Lecture Notes in Mathematics, Vol. 888, Springer-Verlag, New York, 1981.
3. A. A. GONČAR AND G. LÓPES, On Markov's theorem for multipoint Padé approximants for functions of Stieltjes type, *Mat. Sb.* **105** (1978), 512–524; English translation, *Math. USSR-Sb.* **34** (1978), 449–459.
4. P. GONZÁLEZ VERA AND O. NJÅSTAD, Convergence of two-point Padé approximants to series of Stieltjes, *J. Comput. Appl. Math.* **32** (1990), 97–105.
5. G. H. HARDY, J. E. LITTLEWOOD, AND G. POLYA, "Inequalities," Cambridge Univ. Press, London/New York, 1959.
6. E. HENDRIKSEN AND O. NJÅSTAD, A Favard theorem for rational functions, *J. Math. Anal. Appl.* **142** (1989), 508–520.
7. W. B. JONES AND W. J. THRON, "Continued Fractions: Analytic Theory and Applications," Addison-Wesley, Reading, MA, 1980. [Now distributed by Cambridge Univ. Press]
8. J. KARLSSON, Rational interpolation and best rational approximation, *J. Math. Anal. Appl.* **52** (1976), 38–52.
9. J. KARLSSON AND B. VON SYDOW, The convergence of Padé approximants to series of Stieltjes, *Arch. Math.* **14** (1976), 42–53.
10. G. LÓPES, Conditions for convergence of multipoint Padé approximants for functions of Stieltjes type, *Math. Sb.* **107** (1978), 69–83; English translation, *Math. USSR-Sb.* **35** (1979), 363–376.
11. G. LÓPES, Survey on multipoint Padé approximation to Markov type meromorphic functions and asymptotic properties of the orthogonal polynomials generated by them, *in* "Polynômes Orthogonaux et Applications" (C. Brezinski, A. Draux, A. P. Magnus, P. Maroni, et A. Ronveaux, Eds.), pp. 309–316, Lecture Notes in Mathematics, Vol. 1171, Springer-Verlag, New York/Berlin, 1985.
12. G. LÓPES, On the asymptotics of the radio of orthogonal polynomials and convergence of multipoint Padé approximants, *Mat. Sb.* **128** (1985), 216–229; English translation, *Math. USSR-Sb.* **56** (1987), 207–219.
13. O. NJÅSTAD, An extended Hamburger moment problem, *Proc. Edinburgh Math. Soc. (2)* **28** (1985), 167–183.
14. O. NJÅSTAD, Unique solvability of an extended Hamburger moment problem, *J. Math. Anal. Appl.* **124** (1987), 502–519.
15. O. NJÅSTAD, Multipoint Padé approximation and orthogonal rational functions, *in* "Nonlinear Numerical Methods in Rational Approximation" (A. Cuyt, Ed.), pp. 259–270, Reidel, Dordrecht, 1988.
16. O. NJÅSTAD, Orthogonal rational functions with poles in a finite subset of **R**, *in* "Orthogonal Polynomials and Their Applications" (M. Alfaro, J. S. Dehesa, F. J. Marcellan, J. L. Rubio de Francia, and J. Vinuesa, Eds.), pp. 300–307, Lecture Notes in Mathematics, Vol. 1329, Springer-Verlag, New York/Berlin, 1988.
17. O. NJÅSTAD, Unique solvability of an extended Stieltjes moment problem, *Proc. Amer. Math. Soc.* **102** (1988), 78–82.
18. E. B. SAFF, An extension of Montessus de Ballore's theorem on the convergence of interpolating rational functions, *J. Approx. Theory* **6** (1972), 63–67.
19. E. C. TITCHMARSH, "The Theory of Functions," Oxford Univ. Press, London/New York, 1939.
20. H. WALLIN, Rational interpolation to meromorphic functions, *in* "Padé Approximation and Its Applications" (M. G. de Bruin and H. van Rossum, Eds.), pp. 371–382, Lecture Notes in Mathematics, Vol. 888, Springer-Verlag, New York/Berlin, 1981.